Back paper

Number Theory

Instructor: Ramdin Mawia | Marks: **45** | Course: M1 | **Time:** June 7, 2024; 10:00–13:00.

Attempt any FIVE problems. Each question carries 10 marks. The maximum you can score is 45

1. Solve the congruence $3x^2 + 4x + 7 \equiv 0 \pmod{169}$, or prove that it has no solutions in integers. **10**
2. Find the last 2024 digits of $7^{10^{2023}}$. **10**
3. Describe all primes p for which 2 is a quadratic residue mod p. **10**
4. Prove that **10**

$$\sum_{n \leqslant x} \frac{\tau(n)}{n^\alpha} = O(1)$$

   for $\alpha > 1$. Here $\tau(n)$ denotes the divisor-counting function.

5. Let $p \equiv 1 \pmod 3$ be a prime. Show that there are integers x and y such that $x^2 + 3y^2 = 4p$. **10**
6. Let $d \neq 1$ be a squarefree integer, and let $K = \mathbb{Q}[\sqrt{d}]$. Prove that the ring of integers $\mathcal{O}_K$ in K is either $\mathbb{Z}[\sqrt{d}]$ or $\mathbb{Z}[(1+\sqrt{d})/2]$ depending on whether $d \not\equiv 1 \pmod 4$ or $d \equiv 1 \pmod{4.}$ **10**